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THE OMNIBUS TEST STATISTICS USING SRIVASTAVA'S SKEWNESS AND KURTOSIS

RIE ENOMOTO

Department of Mathematical Information Science
Graduate School of Science
Tokyo University of Science
1-3, Kagurazaka, Shinjuku-ku
Tokyo, 162-8601
Japan
e-mail: j1410701@yahoo.co.jp

Abstract

Koizumi et al. [5] proposed the omnibus test statistics using the sample skewness and kurtosis defined by Mardia [7] and Srivastava [16]. However, none of these statistics account for the covariance between skewness and kurtosis, which is not negligible for small sample size. In this paper, we consider the multivariate normality tests based on the sample measures of multivariate skewness and kurtosis defined by Srivastava [16]. We propose some new test statistics, which are taken into consideration the covariance between skewness and kurtosis. In order to evaluate accuracy of proposed test statistics, the numerical results by Monte Carlo simulation for some selected values of parameters are presented.

1. Introduction

For univariate sample case, the test statistic using order statistic derived by Shapiro and Wilk [15] is one of the most famous and essential

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tests for normality. Multivariate extensions of the Shapiro-Wilk test were proposed by Malkovich and Afifi [6], Royston [13], Srivastava and Hui [17] and so on. Another approach for testing normality uses sample skewness and kurtosis separately. Jarque and Bera [4] proposed the bivariate test by using univariate sample skewness and kurtosis. The improved Jarque-Bera (JB) test statistics have been considered by many authors (see, e.g., Urzúa [18] and Nakagawa et al. [10]).

Definitions of multivariate skewness and kurtosis are considered by many authors (see, e.g., Mardia [7, 8], Isogai [3], Oja [11], and Srivastava [16]). Mardia and Foster [9] proposed the omnibus test statistics by using Mardia's sample skewness and kurtosis, which are considered the covariance between skewness and kurtosis. The multivariate JB (MJB type) test statistics using Srivastava's sample skewness and kurtosis that are asymptotically distributed as χ^2-distribution were proposed by Koizumi et al. [5]. Enomoto et al. [2] proposed an improved test statistic by taking account of the variance of test statistic proposed by Koizumi et al. [5]. Furthermore, for small sample size, the test statistic proposed by Enomoto et al. [2] is improved accuracy about the upper percentiles, but the shape of the distribution is not improved. Additionally, none of these statistics account for the covariance between skewness and kurtosis, which is not negligible.

In this paper, we consider the multivariate normality tests based on the sample measures of multivariate skewness and kurtosis defined by Srivastava [16]. We propose some new test statistics, which are taken into consideration the covariance between skewness and kurtosis. We give the numerical results by Monte Carlo simulation for some selected values of parameters in order to evaluate accuracy of proposed test statistics.

2. Srivastava's Measures of Multivariate Skewness and Kurtosis

Let $\boldsymbol{x}$ be a p-dimensional random vector with mean vector $\boldsymbol{\mu}$ and covariance matrix $\Sigma = \Gamma D_\lambda \Gamma'$, where $\Gamma = (\boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}, \ldots, \boldsymbol{\gamma_p})$ is an orthogonal matrix and $D_{\lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$. Note that $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of Σ. Then, Srivastava [16] defined the population measures of multivariate skewness and kurtosis as

$$\beta_{1,p}^2 = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{E[(y_i - \theta_i)^3]}{\lambda_i^{\frac{3}{2}}}\right\}^2,$$

$$\beta_{2,p} = \frac{1}{p}\sum_{i=1}^{p}\frac{E[(y_i - \theta_i)^4]}{\lambda_i^2},$$

respectively, where $y_i = \boldsymbol{\gamma}_i'\boldsymbol{x}$ and $\theta_i = \boldsymbol{\gamma}_i'\boldsymbol{\mu}$ $(i = 1, 2, \ldots, p)$. We note that $\beta_{1,p}^2 = 0$, $\beta_{2,p} = 3$ under a multivariate normal population.

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be samples of size N from a multivariate population. Let $\overline{\boldsymbol{x}}$ and $S = HD_{\omega}H'$ be the sample mean vector and sample covariance matrix given as

$$\overline{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{x}_j,$$

$$S = \frac{1}{N}\sum_{j=1}^{N}(\boldsymbol{x}_j - \overline{\boldsymbol{x}})(\boldsymbol{x}_j - \overline{\boldsymbol{x}})',$$

respectively, where $H = (\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_p)$ is an orthogonal matrix and $D_{\omega} = \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_p)$. We note that

$$\omega_i = \boldsymbol{h}_i'S\boldsymbol{h}_i = \frac{1}{N}\sum_{j=1}^{N}(y_{ij} - \overline{y}_i)^2, \quad i = 1, 2, \ldots, p,$$

where $y_{ij} = \boldsymbol{h}_i'\boldsymbol{x}_j$ $(i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, N)$, $\overline{y}_i = N^{-1}\sum_{j=1}^{N} y_{ij}$ $(i = 1, 2, \ldots, p)$. Then, Srivastava [16] defined the sample measures of multivariate skewness and kurtosis as

$$b_{1,p}^2 = \frac{1}{p}\sum_{i=1}^{p}\left\{\frac{1}{\omega_i^{\frac{3}{2}}}\sum_{j=1}^{N}\frac{(y_{ij}-\bar{y}_i)^3}{N}\right\}^2,$$

$$b_{2,p} = \frac{1}{p}\sum_{i=1}^{p}\frac{1}{\omega_i^2}\sum_{j=1}^{N}\frac{(y_{ij}-\bar{y}_i)^4}{N},$$

respectively.

3. Omnibus Test Statistics Using Multivariate Skewness and Kurtosis

Srivastava [16] and Koizumi et al. [5] show that the sample skewness and kurtosis are distributed as

$$\frac{pb_{1,p}^2}{E[b_{1,p}^2]} \xrightarrow{d} \chi_p^2, \qquad \frac{b_{2,p}-E[b_{2,p}]}{\sqrt{\mathrm{Var}[b_{2,p}]}} \xrightarrow{d} N(0, 1),$$

respectively. Moreover, for large N, Srivastava [16] obtained the expectation of $b_{1,p}^2$ and the expectation and variance of $b_{2,p}$ as

$$E[b_{1,p}^2] = \frac{6}{N}, \quad E[b_{2,p}] = 3, \quad \mathrm{Var}[b_{2,p}] = \frac{24}{pN},$$

respectively. Okamoto and Seo [12] derived the expectation of $b_{1,p}^2$, and Seo and Ariga [14] derived the expectation and variance of $b_{2,p}$ as

$$E[b_{1,p}^2] = \frac{6(N-2)}{(N+1)(N+3)},$$

$$E[b_{2,p}] = \frac{3(N-1)}{N+1}, \quad \mathrm{Var}[b_{2,p}] = \frac{24N(N-2)(N-3)}{p(N+1)^2(N+3)(N+5)},$$

respectively. Koizumi et al. [5] proposed test statistic as

$$MJB = Np\left\{\frac{b_{1,p}^2}{6} + \frac{(b_{2,p}-3)^2}{24}\right\} \xrightarrow{d} \chi_{p+1}^2. \tag{1}$$

(1) is not considered the covariance between skewness and kurtosis. Thus, in this section, we propose Wald type test statistics, which are taken into consideration the covariance between skewness and kurtosis, and also propose MJB type test statistics. Enomoto et al. [2] obtained the covariance between $b_{1,p}^2$ and $b_{2,p}$ as

$$\mathrm{Cov}[b_{1,p}^2, b_{2,p}] = \frac{216N(N-2)(N-3)}{p(N+1)^2(N+3)(N+5)(N+7)}, \quad N \neq 2, 3.$$

Furthermore, we give the correlation of $b_{1,p}^2$ and $b_{2,p}$ by using the same way as Enomoto et al. [2] as follows:

$$\mathrm{Corr}[b_{1,p}^2, b_{2,p}] = \sqrt{\frac{27(N-3)(N+3)(N+9)}{N^2(N+7)(N^3+37N^2+11N-313)}}.$$

Since $p(N+1)(N+3)b_{1,p}^2/\{6(N-2)\}$ is distributed as χ_p^2-distribution, we obtain the expectation and variance of $\sqrt{b_{1,p}^2}$ as follows:

$$E\left[\sqrt{b_{1,p}^2}\right] = \sqrt{\frac{6(N-2)}{p(N+1)(N+3)}} \times \frac{\sqrt{2}\Gamma\left[\frac{p+1}{2}\right]}{\Gamma\left[\frac{p}{2}\right]},$$

$$\mathrm{Var}\left[\sqrt{b_{1,p}^2}\right] = \frac{6(N-2)}{p(N+1)(N+3)} \times \left(p - \frac{2\left\{\Gamma\left[\frac{p+1}{2}\right]\right\}^2}{\left\{\Gamma\left[\frac{p}{2}\right]\right\}^2}\right).$$

Further, we obtain the covariance between $\sqrt{b_{1,p}^2}$ and $b_{2,p}$ by using a Taylor series expansion as

$$\mathrm{Cov}\left[\sqrt{b_{1,p}^2}, b_{2,p}\right] \approx \frac{1}{2\sqrt{E[b_{1,p}^2]}} \mathrm{Cov}\left[b_{1,p}^2, b_{2,p}\right]$$

$$= \frac{18N(N-3)}{p(N+1)(N+5)(N+7)}\sqrt{\frac{6(N-2)}{(N+1)(N+3)}} + O(N^{-2}).$$

Therefore, we propose the Wald type test statistic as

$$c_R = \boldsymbol{b}_R' V_R^{-1} \boldsymbol{b}_R, \tag{2}$$

where

$$\boldsymbol{b}_R = \left\{ \sqrt{b_{1,p}^2} - E\left[\sqrt{b_{1,p}^2} \right], \; b_{2,p} - \frac{3(N-1)}{N+1} \right\}',$$

and

$$V_R = \begin{pmatrix} \mathrm{Var}\left[\sqrt{b_{1,p}^2} \right] & \mathrm{Cov}\left[\sqrt{b_{1,p}^2}, \, b_{2,p} \right] \\ \mathrm{Cov}\left[\sqrt{b_{1,p}^2}, \, b_{2,p} \right] & \dfrac{24}{pN} \end{pmatrix}.$$

Also, the test statistics for sample skewness using the Wilson-Hilferty transformation or the central limit theorem are obtained as

$$W(b_{1,p}^2) = \left[\left\{ \frac{(N+1)(N+3)}{6(N-2)} b_{1,p}^2 \right\}^{\frac{1}{3}} + \frac{2}{9p} - 1 \right] \left(\frac{2}{9p} \right)^{-\frac{1}{2}} \xrightarrow{d} N(0, 1),$$

$$U(b_{1,p}^2) = \frac{b_{1,p}^2 - \dfrac{6}{N}}{\dfrac{6\sqrt{2p}}{pN}} \xrightarrow{d} N(0, 1).$$

Seo and Ariga [14] proposed the normalizing transformation test statistic for sample kurtosis as

$$z_{NT} = \sqrt{\frac{Np}{24}} \left\{ -e^{-b_{2,p}+3} + \frac{6(p+2)}{Np} + 1 \right\} \xrightarrow{d} N(0, 1).$$

Therefore, we propose the MJB type test statistics using $W(b_{1,p}^2)$ or $U(b_{1,p}^2)$ and z_{NT} as

$$W_{NT} = W(b_{1,p}^2)^2 + z_{NT}^2, \tag{3}$$

$$S_{NT} = U(b_{1,p}^2)^2 + z_{NT}^2. \tag{4}$$

Moreover, we obtain the covariance between $W(b_{1,p}^2)$ and z_{NT} by using a Taylor series expansion as

$$\mathrm{Cov}\,[W(b_{1,p}^2),\, z_{NT}] \approx \frac{p(N+1)(N+3)}{24(N-2)}\sqrt{\frac{N}{3}}\cdot\left\{\frac{N(N-2)}{(N+1)(N+3)}\right\}^{\frac{2}{3}}\mathrm{Cov}\,[b_{1,p}^2,\, b_{2,p}]$$

$$= \frac{3\sqrt{3}N^{\frac{3}{2}}(N-3)}{(N+1)(N+5)(N+7)}\cdot\left\{\frac{N(N-2)}{(N+1)(N+3)}\right\}^{\frac{2}{3}} + O(N^{-2}).$$

Then, we propose the Wald type test statistic as

$$c_W = \boldsymbol{b}_W' V_W^{-1} \boldsymbol{b}_W, \tag{5}$$

where

$$\boldsymbol{b}_W = \{W(b_{1,p}^2),\, z_{NT}\}',$$

and

$$V_W = \begin{pmatrix} 1 & \mathrm{Cov}\,[W(b_{1,p}^2),\, z_{NT}] \\ \mathrm{Cov}\,[W(b_{1,p}^2),\, z_{NT}] & 1 \end{pmatrix}.$$

Another test statistic for sample skewness using the central limit theorem is obtained as

$$U^*(b_{1,p}^2) = \frac{b_{1,p}^2 - \dfrac{6(N-2)}{(N+1)(N+3)}}{\dfrac{6\sqrt{2p}(N-2)}{p(N+1)(N+3)}} \xrightarrow{d} N(0,\,1).$$

For sample kurtosis, we propose the standardized test statistic $U(b_{2,p})$, and Seo and Ariga [14] proposed the standardized test statistic $U^*(b_{2,p})$ as

$$U(b_{2,p}) = \sqrt{\frac{pN}{24}}\left\{b_{2,p} - \frac{3(N-1)}{N+1}\right\} \xrightarrow{d} N(0,\,1),$$

$$U^*(b_{2,p}) = \sqrt{\frac{p(N+1)^2(N+3)(N+5)}{24N(N-2)(N-3)}}\left\{b_{2,p} - \frac{3(N-1)}{N+1}\right\} \xrightarrow{d} N(0, 1),$$

respectively. Then, we propose MJB type test statistics using $U^*(b_{1,p}^2)$ and $U^*(b_{2,p})$ or $U(b_{2,p})$ as

$$S_N = U^*(b_{1,p}^2)^2 + U^*(b_{2,p})^2, \tag{6}$$

$$S_N^* = U^*(b_{1,p}^2)^2 + U(b_{2,p})^2, \tag{7}$$

and we propose the Wald type test statistics as

$$c_N = \boldsymbol{b}_N' V_N^{-1} \boldsymbol{b}_N, \tag{8}$$

$$c_N^* = \boldsymbol{b}_N' V_N^{*-1} \boldsymbol{b}_N, \tag{9}$$

where

$$\boldsymbol{b}_N = \left\{b_{1,p}^2 - \frac{6(N-2)}{(N+1)(N+3)},\ b_{2,p} - \frac{3(N-1)}{N+1}\right\}',$$

$$V_N = \begin{pmatrix} \frac{2}{p}\left\{\frac{6(N-2)}{(N+1)(N+3)}\right\}^2 & \mathrm{Cov}\left[b_{1,p}^2,\ b_{2,p}\right] \\ \mathrm{Cov}\left[b_{1,p}^2,\ b_{2,p}\right] & \frac{24N(N-2)(N-3)}{p(N+1)^2(N+3)(N+5)} \end{pmatrix},$$

and

$$V_N^* = \begin{pmatrix} \frac{2}{p}\left\{\frac{6(N-2)}{(N+1)(N+3)}\right\}^2 & \mathrm{Cov}\left[b_{1,p}^2,\ b_{2,p}\right] \\ \mathrm{Cov}\left[b_{1,p}^2,\ b_{2,p}\right] & \frac{24}{pN} \end{pmatrix}.$$

The Wald type test statistics and MJB type test statistics (2)-(9) are distributed as χ_2^2 distribution under the normality.

4. Simulation Studies

The accuracies of upper percentiles, type I errors, powers and relative errors for upper percentiles of (1)-(9) are evaluated via a Monte Carlo simulation study, where (2)-(9) are derived in this paper and (1) is proposed by Koizumi et al. [5]. Simulation parameters are as follows: $p = 3, 10, 20, 30$, $N = 20, 50, 100, 200, 400, 800$ $(p < N)$; and significance level $\alpha = 0.05$. As a numerical experiment, we carry out 1,000,000 replications.

Table 1 gives the values of the upper 5 percentiles of (1)-(9). The lowest row of tables shows theoretical values for the χ^2-distributions. When the number of samples is about 50, (2) is closer to the χ^2-distribution rather than (3)-(9) for all dimensions. We confirm the following tendency about the test statistics (3)-(9). When p is small, the upper percentiles of (3) are closer to the χ^2-distribution rather than (4)-(9). When the number of dimensions is about 10, the Wald type test statistic (5) has good accuracy rather than MJB type test statistic (3). (9) converges in the χ^2-distribution when the number of dimensions is about 20. For large p, the Wald type test statistic (9) has good accuracy rather than MJB type test statistic (7). Similarly, the Wald type test statistic (8) has good accuracy rather than MJB type test statistic (6) when p is large. For MJB type test statistics (6) and (7), (7) has good accuracy rather than (6). For the Wald type test statistics (8) and (9), (9) has good accuracy rather than (8) when the number of dimensions is about 10. (4) converges in the χ^2-distribution when sample size increases. For (3) and (4), (4) has good accuracy rather than (3) for many parameters. In order to obtain the values of relative errors for upper percentiles in Table 5, the upper percentiles for (1) is listed.

Table 1. Upper 5 percentiles

p	N	c_R (2)	W_{NT} (3)	S_{NT} (4)	c_W (5)	S_N (6)	S_N^* (7)	c_N (8)	c_N^* (9)	MJB (1)
3	20	5.004	6.153	4.728	5.209	7.834	6.214	9.658	5.571	6.801
	50	5.909	6.048	5.661	5.698	7.330	6.457	8.007	3.880	8.445
	100	5.989	5.927	5.990	5.803	6.946	6.520	7.245	6.618	8.994
	200	5.955	5.889	6.193	5.813	6.712	6.481	6.853	4.276	9.263
	400	5.936	5.858	6.286	5.815	6.557	6.482	6.616	6.480	9.376
	800	5.935	5.843	6.374	5.825	6.524	6.440	6.548	6.481	9.468
	∞	5.992								9.488
10	20	5.334	7.200	5.626	5.740	7.924	6.121	9.165	5.398	15.00
	50	5.912	6.737	5.900	6.083	7.116	6.278	7.485	6.065	17.89
	100	5.981	6.447	6.048	6.120	6.710	6.227	6.770	6.163	18.87
	200	5.973	6.230	6.107	6.092	6.449	6.217	6.435	6.158	19.33
	400	5.980	6.106	6.113	6.023	6.298	6.176	6.287	6.156	19.53
	800	5.969	6.041	6.116	6.008	6.213	6.151	6.191	6.126	19.63
	∞	5.992								19.68
20	50	5.969	7.007	6.200	6.200	7.236	6.368	7.324	5.974	29.81
	100	6.016	6.555	6.098	6.222	6.693	6.263	6.638	6.054	31.32
	200	5.994	6.294	6.062	6.137	6.376	6.153	6.294	6.029	32.03
	400	6.001	6.155	6.050	6.077	6.214	6.116	6.168	6.036	32.37
	800	5.975	6.060	6.018	6.031	6.105	6.061	6.078	6.016	32.53
	∞	5.992								32.67
30	50	6.005	7.117	6.441	6.286	7.329	6.472	7.313	5.993	41.10
	100	6.031	6.629	6.183	6.245	6.734	6.299	6.603	6.039	43.13
	200	6.015	6.337	6.107	6.140	6.410	6.158	6.288	6.030	44.12
	400	5.999	6.161	6.043	6.068	6.199	6.080	6.135	6.008	44.55
	800	5.987	6.081	6.024	6.031	6.104	6.040	6.067	6.007	44.75
	∞	5.992								44.99

Table 2 gives the values for type I errors of (1)-(9). The values of (9) are closer to 0.05 when p is large. The values of (2) and (4) are the best for many parameters. Furthermore, (2) and (4) are closer to 0.05 when the number of samples is about 50. When sample size is small, the test statistics except (8) are closer to 0.05 rather than (1). When the number of dimensions is about 10, (5) has good accuracy rather than (3). (6) is closer to 0.05 rather than (8) for many parameters. When p and N increase, (9) has good accuracy rather than (7). For (6) and (7), (7) has good accuracy rather than (6). For (8) and (9), (9) has good accuracy rather than (8).

Table 2. Type I errors

p	N	c_R (2)	W_{NT} (3)	S_{NT} (4)	c_W (5)	S_N (6)	S_N^* (7)	c_N (8)	c_N^* (9)	MJB (1)
3	20	0.033	0.053	0.032	0.033	0.066	0.052	0.103	0.046	0.021
	50	0.049	0.051	0.045	0.043	0.063	0.054	0.078	0.056	0.037
	100	0.050	0.049	0.050	0.045	0.061	0.056	0.068	0.058	0.043
	200	0.049	0.048	0.053	0.045	0.060	0.056	0.063	0.058	0.046
	400	0.049	0.047	0.055	0.045	0.058	0.057	0.060	0.058	0.048
	800	0.049	0.046	0.056	0.046	0.059	0.057	0.059	0.058	0.050
10	20	0.038	0.076	0.043	0.044	0.077	0.051	0.110	0.043	0.013
	50	0.048	0.066	0.048	0.052	0.067	0.054	0.077	0.051	0.032
	100	0.050	0.060	0.051	0.053	0.062	0.054	0.065	0.053	0.041
	200	0.050	0.056	0.052	0.053	0.058	0.054	0.058	0.053	0.046
	400	0.050	0.053	0.052	0.051	0.056	0.053	0.056	0.053	0.048
	800	0.049	0.051	0.052	0.050	0.054	0.053	0.054	0.053	0.049
20	50	0.049	0.072	0.054	0.055	0.072	0.056	0.078	0.050	0.027
	100	0.051	0.063	0.052	0.056	0.063	0.055	0.064	0.051	0.037
	200	0.050	0.057	0.052	0.054	0.058	0.053	0.057	0.051	0.043
	400	0.050	0.054	0.051	0.052	0.055	0.053	0.054	0.051	0.047
	800	0.050	0.052	0.051	0.051	0.052	0.051	0.052	0.051	0.048
30	50	0.050	0.074	0.060	0.057	0.076	0.059	0.079	0.050	0.023
	100	0.051	0.065	0.054	0.056	0.065	0.056	0.064	0.051	0.035
	200	0.051	0.058	0.053	0.054	0.059	0.053	0.057	0.051	0.042
	400	0.050	0.054	0.051	0.052	0.055	0.052	0.053	0.050	0.046
	800	0.050	0.052	0.051	0.051	0.052	0.051	0.052	0.050	0.048
	∞	0.050								

Table 3 gives the values of the powers for (1)-(9). We performed simulations under multivariate t-distribution with 5 degrees of freedom. Test statistics (2)-(9) are more powerful rather than (1) for many parameters. (3) is more powerful rather than (5). For many parameters, (8) is more powerful rather than (6). Although the values of power for (2) are not the best, its values have powers comparable to (3)-(9).

Table 3. Powers under multivariate t-distribution

p	N	c_R (2)	W_{NT} (3)	S_{NT} (4)	c_W (5)	S_N (6)	S_N^* (7)	c_N (8)	c_N^* (9)	MJB (1)
3	20	0.299	0.312	0.274	0.248	0.408	0.358	0.456	0.335	0.267
	50	0.681	0.604	0.595	0.550	0.728	0.693	0.748	0.696	0.646
	100	0.920	0.873	0.869	0.860	0.929	0.920	0.935	0.924	0.897
	200	0.996	0.992	0.992	0.992	0.996	0.996	0.997	0.996	0.993
	400	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
10	20	0.507	0.562	0.477	0.489	0.642	0.583	0.657	0.544	0.435
	50	0.922	0.918	0.909	0.899	0.947	0.932	0.949	0.928	0.880
	100	0.998	0.998	0.998	0.998	0.999	0.999	0.999	0.999	0.995
	200	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	400	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
20	50	0.980	0.982	0.979	0.977	0.998	0.984	0.988	0.982	0.952
	100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	200	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	400	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
30	50	0.995	0.995	0.994	0.994	0.997	0.996	0.997	0.995	0.979
	100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	200	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	400	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 4 gives the values of the powers for (1)-(9), where each element of the sample is generated using χ_5^2-distribution. Test statistics (2)-(9) are more powerful rather than (1) for many parameters. (8) is more powerful rather than (6) for small p and N. (3) is more powerful rather than (5). Although the values of power for (2) are not the best, its values have powers comparable to (3)-(9).

Table 4. Powers under χ^2-distribution

p	N	c_R (2)	W_{NT} (3)	S_{NT} (4)	c_W (5)	S_N (6)	S_N^* (7)	c_N (8)	c_N^* (9)	MJB (1)
3	20	0.334	0.328	0.298	0.263	0.414	0.394	0.469	0.395	0.264
	50	0.799	0.730	0.770	0.709	0.809	0.805	0.828	0.814	0.756
	100	0.956	0.945	0.962	0.937	0.968	0.967	0.968	0.966	0.960
	200	0.997	0.996	0.998	0.995	0.998	0.998	0.998	0.998	0.998
	400	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
10	20	0.221	0.258	0.178	0.198	0.322	0.287	0.344	0.267	0.154
	50	0.574	0.593	0.583	0.531	0.663	0.647	0.648	0.625	0.586
	100	0.878	0.888	0.898	0.856	0.919	0.914	0.905	0.900	0.903
	200	0.992	0.994	0.995	0.990	0.996	0.996	0.995	0.994	0.995
	400	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
20	50	0.395	0.436	0.394	0.370	0.498	0.473	0.471	0.436	0.386
	100	0.699	0.736	0.733	0.675	0.781	0.769	0.746	0.733	0.729
	200	0.954	0.964	0.966	0.948	0.973	0.971	0.964	0.962	0.964
	400	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000
	800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
30	50	0.309	0.356	0.299	0.293	0.410	0.382	0.382	0.342	0.278
	100	0.548	0.597	0.580	0.527	0.647	0.630	0.600	0.581	0.563
	200	0.871	0.895	0.896	0.859	0.914	0.998	0.891	0.885	0.879
	400	0.996	0.997	0.997	0.995	0.998	1.000	0.996	0.996	0.995
	800	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

It is note that a multivariate t-distribution is symmetric and has heavy tails rather than a multivariate normal distribution. Further, a χ^2-distribution has asymmetric and its kurtosis also differs from a normal distribution. It seem that the difference between each power do not have a distinctive feature.

Table 5 gives relative errors for the upper percentiles of (1)-(9). When N is small, there is difference between (1) and χ^2_{p+1}-distribution. The Wald type test statistic (2) has good accuracy rather than (1) for all parameters. For N and p are small, (2)-(9) have good accuracy rather than (1). When the number of dimensions is about 20, (9) has good accuracy rather than (1). For many parameters, (5) has good accuracy rather than (1).

In conclusion, most of test statistics (2)-(9) have good accuracy rather than (1) for many parameters. Peculiarly, the Wald type test statistic (2) is the best at the point of the upper percentiles, type I errors and powers. When the number of dimensions is about 10, the Wald type test statistic (5) has good accuracy rather than MJB type test statistic (3). Furthermore, (2) and (5) have good shapes of their distributions even for small sample size. However, other test statistics do not have good shapes of their distributions when sample size is small. For large p, the Wald type test statistic (9) has also good accuracy rather than (3)-(8) and (1).

Table 5. Relative errors for the upper percentiles

p	N	c_R (2)	W_{NT} (3)	S_{NT} (4)	c_W (5)	S_N (6)	S_N^* (7)	c_N (8)	c_N^* (9)	MJB (1)
3	20	– 0.197	0.026	– 0.267	– 0.150	0.235	0.036	0.380	– 0.076	– 0.395
	50	– 0.014	0.009	– 0.058	– 0.051	0.183	0.072	0.252	– 0.544	– 0.124
	100	0.000	– 0.011	0.000	– 0.033	0.137	0.081	0.173	0.095	– 0.055
	200	– 0.006	– 0.017	0.032	– 0.031	0.107	0.075	0.126	– 0.401	– 0.024
	400	– 0.009	– 0.023	0.047	– 0.030	0.086	0.076	0.094	0.075	– 0.012
	800	– 0.010	– 0.025	0.060	– 0.029	0.082	0.070	0.085	0.076	– 0.002
10	20	– 0.123	0.168	– 0.065	– 0.044	0.244	0.021	0.346	– 0.110	– 0.312
	50	– 0.014	0.111	– 0.015	0.015	0.158	0.046	0.200	0.012	– 0.100
	100	– 0.002	0.071	0.009	0.021	0.107	0.038	0.115	0.028	– 0.043
	200	– 0.003	0.038	0.019	0.017	0.071	0.036	0.069	0.027	– 0.018
	400	– 0.002	0.019	0.020	0.005	0.049	0.030	0.047	0.027	– 0.007
	800	– 0.004	0.008	0.020	0.003	0.036	0.026	0.032	0.022	– 0.003
20	50	– 0.004	0.145	0.034	0.034	0.172	0.059	0.182	– 0.003	– 0.096
	100	0.004	0.086	0.017	0.037	0.105	0.043	0.097	0.010	– 0.043
	200	0.000	0.048	0.012	0.024	0.060	0.026	0.048	0.006	– 0.020
	400	0.002	0.027	0.010	0.014	0.036	0.020	0.029	0.007	– 0.009
	800	– 0.003	0.011	0.004	0.007	0.019	0.011	0.014	0.004	– 0.004
30	50	0.002	0.158	0.070	0.047	0.182	0.074	0.181	0.000	– 0.094
	100	0.007	0.096	0.031	0.041	0.110	0.049	0.093	0.008	– 0.043
	200	0.004	0.055	0.019	0.024	0.065	0.027	0.047	0.006	– 0.020
	400	0.001	0.028	0.008	0.013	0.033	0.015	0.023	0.003	– 0.010
	800	– 0.001	0.015	0.005	0.007	0.018	0.008	0.012	0.003	– 0.005

5. An Example for Multivariate Normality Tests

We illustrate the multivariate normality tests with real data given by Brunner et al. [1]. Table 6 presents the values of γ -GT for the 26 patients. The 26 patients whose gall bladders had to be removes in consequence of a cholelithiasis (without a blockage of the ductus choledochus) were selected to take part in a randomized study in which 26 of the patients were treated with a specific drug. The γ -GT of each patient was determined before the operation (–1) and on days 3, 7, and 10 after the operation. The efficacy of the drug is represented by the different profiles of the γ-GT, that is, the existence of an interaction between treatment and time.

Table 6. The postoperative level of γ-GT for the 26 patients with a specific drug

	Day after surgery					Day after surgery			
Patient	– 1	3	7	10	Patient	– 1	3	7	10
1	44	12	10	9	26	50	30	29	35
5	15	14	14	15	27	13	21	29	15
6	8	10	9	9	29	7	8	7	9
8	12	17	28	31	33	7	14	25	19
9	7	26	29	22	36	11	11	12	15
11	8	10	9	12	37	192	157	92	66
12	32	226	118	76	38	14	12	20	16
18	109	104	66	48	42	24	9	10	12
14	53	49	50	49	44	9	14	16	13
17	56	162	111	79	45	16	32	28	20
21	11	15	26	12	47	9	13	12	13
24	38	100	47	67	49	19	14	13	12
25	13	167	139	110	50	8	10	10	11
					Mean	30.19	48.35	36.89	30.58

Table 7 gives the results for multivariate normality tests, where significance level $\alpha = 0.05$. For original data, we obtain that both p-values are zero. Because each p-value is smaller than 0.05, the null hypothesis of normality is rejected. Frequently, it is thought that the pharmaceutical data is distributed as a log-normal distribution. Therefore, the null hypothesis is accepted when we test after logarithm transformation. That is, from results of normality tests, we cannot assume that original data is distributed as a normal distribution. When analyzing this data, we need to transformation or nonparametric analysis on the data.

Table 7. The results for multivariate normality tests

Original data									
	c_R	W_{NT}	S_{NT}	c_W	S_N	S_N^*	c_N	c_N^*	MJB
	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(1)
Test statistic	101.4	44.2	190.9	36.5	466.4	395.1	301.1	300.9	127.8
p-value	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Data of logarithmic transformation									
	c_R	W_{NT}	S_{NT}	c_W	S_N	S_N^*	c_N	c_N^*	MJB
	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(1)
Test statistic	1.31	2.27	1.35	1.53	2.21	1.76	1.26	1.20	5.72
p-value	0.520	0.322	0.510	0.466	0.331	0.414	0.532	0.549	0.335

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